Principle of mathematical induction :—

Mathematical induction is a technique of proving a statement, theorem or formula which is thought to be true, for each and every natural number n. By generalizing this in form of a principle which we would use to prove any mathematical statement is 'principle of mathematical induction'.

For example : $1^3+2^3+3^3+ \cdots +n^3 = (n(n+1)/2)^2$, the statement is considered here as true for all the values of natural numbers.

Principle of mathematical Induction Solution and proof :—

Consider a statement P(n); where n is a natural number. Then to determine the validity of P(n) for every n, use the following principle.

Step-1 :— Check whether the given statement is true for n = 1

Step-2 :— Assume that given statement P(n) is also true for n = k, where k is any positive integer.

Step-3 :— prove that the result is true for P(k+1) for any positive integer k.

If the above-mentioned conditions are satisfied, then it can be concluded that P(n) is true for all n natural numbers.

Proof :-

The first step of the principle is a factual statement and the second step is a conditional one. According to this if the given statement is true for some positive integer k only then it can be concluded that the statement $P(n)$ is valid for $n = k+1$

This is also know as the inductive step and the assumption that $P(n)$ is true for $n = k$ is know as the inductive hypothesis.

Solved problems :-

Q. Prove that the sum of cubes of n natural numbers is equal to $\left(\frac{n(n+1)}{2}\right)^2$ for all n natural numbers.

Soln In the given statement we are asked to prove :

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Step-1 :- Now with the help of the principle of induction in math let us check the validity of the given statement $P(n)$ for $n = 1$.

$$P(1) = \left(\frac{1(1+1)}{2}\right)^2 = 1$$ This is true.

Step-2 :- Now as the given statement is true for $n = 1$ we shall move forward and try proving this for $n = k$, i.e.

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \left(\frac{k(k+1)}{2}\right)^2.$$

Step-3 :- Let us now try to establish that $P(k+1)$ is also true.

$$1^3+2^3+3^3+ \cdots + k^3+(k+1)^3 = \left(k(k+1)^2\right)^2+(k+1)^3$$

$$\Rightarrow 1^3+2^3+3^2+ \cdots +k^3+(k+1)^3 = k^2+(k+1)^4+(k+1)^3$$

$$= k^2(k+1)^2+4((k+1)^3)^4$$

$$=(k+1)^2(k^2+4(k+1))^4$$

$$=(k+1)^2(k^2+4k+4))^4$$

$$=(k+1)^2((k+2)^2)^4$$

$$=(k+1)^2((k+1+1)^2)^4$$

$$=(k+1)^2((k+1)+1)^2)^4$$

Pigeonhole principle :—

In mathematics, the pigeonhole principle states that if n items are put into m containers, with $n > m$, then at least one container must contain more than one item.

Theorem of Arithmetic :—

Theorem of Arithmetic state that every integer greater than 1 is either a prime number or can be expressed in the form of primes.

In other words, all the natural numbers can be expressed in the form of the product of its prime factors. To recall, prime factors are are the numbers which are divisible by 1 and itself only. For example, the number 35 can be written in the form of its prime factor as:

$35 = 7 \times 5$

Here, 7 and 5 are prime factors of 35.

Similarly, another number 114560 can be represented as the product of its prime factors by using prime factorization method.

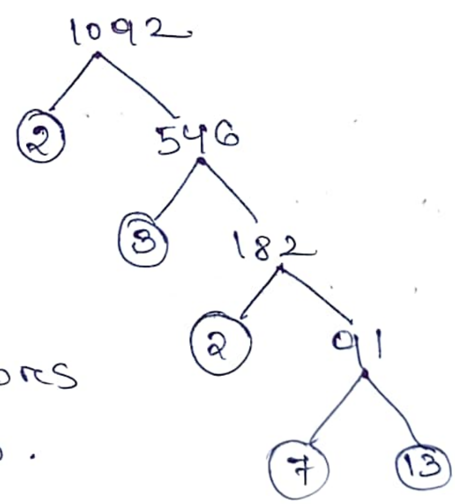
$$114560 = 2^7 \times 5 \times 179$$

So, we have factorized 114560 as the product of the power of its prime.

Check : Prime Factorization method :—

Therefore, every natural number can be expressed in the form of the product of the power of its primes. This statement is know as the Fundamental Theorem of Arithmetic, unique factorization theorem or unique-prime-factorization theorem.

Circle the primes as they appear.



prime factors
2, 3, 2, 7, 13.

Proof In number theory, a composite number is expressed in the form of the product of primes and this factorization is unique apart from the order in which the prime factor occurs.

From this theorem we can also see that not only a composite number can be factorized as the product of their primes but also for each composite number the factorization is unique, not taking into consideration order of occurrence of the prime factors.

In simple words, there exists only a single way to represent a natural number by the product of prime factors. This fact can also be stated as:

The prime factorization of any natural number is said to be unique for except the order of their factors.

In general, a composite number "a" can be expressed as,

$a = P_1 P_2 P_3 \cdots P_n$, where $P_1, P_2, P_3 \cdots P_n$ are the prime factors of a written in ascending order i.e.

$P_1 \leq P_2 \leq P_3 \cdots \leq P_n$.

writing the primes in ascending order makes the factorization unique in nature.

Theorem of Arithmetic Examples: —

Q. In a formula racing competition the time taken by two racing cars A and B to complete 1 round of the track is 30 minutes and 45 minutes respectively. After how much time could the cars meet again at the starting point?

Sol^n. As the time taken by car B is more compared to that of A to complete one round therefore it can be assumed that A will reach early and both the cars will meet again when A has already reached the starting point. This time can be calculated by finding the L.C.M of the time taken by each:

$$30 = 2 \times 3 \times 5$$

$$45 = 3 \times 3 \times 5$$

$\therefore$ The L.C.M is 90.

Thus, both cars will meet at the starting point after 90 minutes.

Permutation And Combination :—

The ways to represent a group of objects by selecting them in a set and forming sub-sets. It defines the various ways to arrange a certain group of data. When we select the data or objects from a certain group, it is said to be permutations, whereas the order in which they are represented is called combination. Both concepts are very important in mathematics.

Permutation and combination are explained here elaborately, along with the difference between them. We will discuss both the topics here with their formulas, real-life examples and solved questions. Students can also work on

Permutation And combination worksheet to enhance their knowledge in this area along with getting tricks to solve more question.

What is permutation ?

In mathematics, permutation relates to the act of arranging all the members or order. In other words, if the set is already ordered, then the rearranging of its elements is called the process of permuting.

Permutations occur, in more or less prominent ways, in almost every area of mathematics. They often arise when different orderings on certain finite sets are considered.

Formula :—

A permutation is the choice of r things from a set of n things without replacement and where the order matters.

$$^{n}P_{r} = (n!) / (n-r)!.$$

What is a Combination ?

The Combination is a way of selecting items from a collection, such that (unlike permutations) the order of selection does not matter. In smaller cases, it is possible to count the number of combinations.

Combination refers to the combination of n things taken k at a time without repetition. To refer to combination in which repetition is allowed, the terms k-selection or k-combination with repetition are often used.

Formula :—

A combination is the choice of r things from a set of n things without replacement and where order does not matter.

$$ {}_nC_r = \binom{n}{r} = \frac{{}_nP_r}{r!} = \frac{n!}{r!\,(n-r)!} $$

Difference Between permutation and Combination :—

Permutation	Combination
1) Arranging people digits, numbers, alphabets, letters and colours.	1) Selection of menu, food, clothes, subjects team.
2) Picking a team Captain, pitcher and Shortstop from a group.	2) Picking three team members from a group.
3) Picking two favourite Colours, in order, from a colour brochure.	3) picking two colours from a colour brochure
4) Picking first, Second and third place winners.	4) Picking three winners

Uses of permutation and Combination :—

A permutation is used for the list of data and the combination is used for a group of data.

Example :—

Find the number of permutations and combinations of $n = 12$ and $r = 2$

Solⁿ Given, $n = 12$

$r = 2$

using the formula given above:

Permutation :—

$^nP_r = (n!)/(n-r)! = (12!)/(12-2)! = 12!/10! = (12 \times 11 \times 10!)/10! = 132.$

Combination :—

$$^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$\Rightarrow \frac{12!}{2!\,(12-2)!} = \frac{12!}{2!\,(10)!} = \frac{12 \times 11 \times 10!}{2!\,(10)!}$$

$= 66.$

Recurrence relation :—

A recurrence relation is an equation that expresses each element of a sequence as a function of the preceding ones. more precisely, in the case where only the immediately preceding element is involved, a recurrence relation has the form.

$$u_n = \varphi(n, u_{n-1}) \text{ for } n > 0,$$

where,

$$\varphi : \mathbb{N} \times X \to X.$$

is a function, where X is set to which the elements of a sequence must belong. For any $u_0 \in X$, this defines a unique sequence with

u_0 as its first element, called the initial value.

It is easy to modify the definition for getting sequences starting from the term of index 1 or higher.

This defines recurrence relation of first order. A recurrence relation of order k has the form

$$u_n = \varphi(n, u_{n-1}, u_{n-2}, \ldots, u_{n-k}) \text{ for } n \geq k,$$

where $\varphi : \mathbb{N} \times X^k \to X$ is a function that involves k consecutive elements of the sequence. In this case, k initial values are needed for defining a sequence.

Generating functions :—

A generating function is a device somewhat similar to a bag. Instead of carrying many little objects detachedly, which could be embarrassing, we put them all in a bag, and then we have only one object to carry, the bag.

George polya, mathematics and plausible reasoning (1954)

A generating function is a clothesline on which we hang up a sequence of numbers for disply.

Herbert wilf, Generatingfunctionology (1994)

Ordinary generating function :— (OGF)

The ordinary generating function of a sequence a_n is

$$G(a_n; x) = \sum_{n=0}^{\infty} a_n x^n .$$

When the term generating function is used without qualification, it is usually taken to mean an ordinary generating function.

If a_n is the probability mass function of a discrete random variable, then its ordinary generating function is called a probability-generating function.

The ordinary generating function can be generalized to arrays with multiple indices. For example, the ordinary generating function of a two-dimensional array $a_{m,n}$ is

$$G(a_{m,n}; x, y) = \sum_{m,n=0}^{\infty} a_{m,n} x^m y^n .$$

Exponential generating function (EGF) :—

The exponential generating function of a sequence a_n is

$$EG(a_n; x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

Exponential generating functions are generally more convenient than ordinary generating function for combinatorial enumeration problems that involve labelled objects.

Poisson generating function :-

The poisson generating function of a sequence a_n is

$$PG(a_n; x) = \sum_{n=0}^{\infty} a_n e^{-x} \frac{x^n}{n!} = e^{-x} \, EG.$$

Example :-

Ordinary generating function :-

$$G(n^2; x) = \sum_{n=0}^{\infty} n^2 x^n = \frac{x(x+1)}{(1-x)^3}$$

Exponential generating function :-

$$EG(n^2; x) = \sum_{n=0}^{\infty} \frac{n^2 x^2}{n!} = x(x+1)e^x.$$