Limit :—

Definition :—

A limit is the value that a function "approaches" as the input "approaches" some value. Limits are essential to calculus and mathematical analysis, and are used to define continuity, derivatives, and integrals.

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x-a}.$$

Example :—

$$\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}$$

Continuity and Differentiability of functions of one variable :—

Continuity at a point, continuity on an interval, derivative of functions and many more. However, continuity and Differentiability of functional parameters are very difficult. Let us take an example to make this simpler.

Consider the function,

$$\begin{cases} x+3 & \text{if } x \le 0 \\ x & \text{if } x > 0 \end{cases}$$

→ For any point on the Real number line, this function is defined.

→ It can be seen that the value of the function $x = 0$ changes suddenly. Following the concepts of limits, we can say that,

→ Right-hand limit ≠ Left-hand limit.

→ It implies that this function is not continuous at $x = 0$.

Definition of Continuity:—

In mathematically, A function is said to be continuous at a point $x = a$, If

$$\lim_{x \to a} f(x) \text{ exists, and.}$$

$$\lim_{x \to a} f(x) = f(a)$$

It implies that if the left hand limit (L.H.L) right hand limit (R.H.L) and the value of the function at $x = a$ exists and these parameters are equal to each other, then the function f is said to be continuous at $x = a$.

If the function is undefined or does not exist. then we say that the function is discontinuous.

Continuity in open interval (a,b) :—

$f(x)$ would be continuous in the open interval (a,b) if at any point in the given interval the function is continuous.

Continuity in closed interval [a, b] :—

A function f(x) is said to be continuous in the closed interval [a,b] if it satisfies the following three conditions.

→ f(x) is be continuous in the open interval (a,b)

→ f(x) is continuous at the point a from right i.e

$$\lim_{x \to a} f(x) = f(a)$$

→ f(x) is continuous at the point b from left i.e.

$$\lim_{x \to b} f(x) = f(b).$$

Example :—

Q. Check the continuity of the function.

$$f(x) = \begin{cases} \dfrac{|x-4|}{x-4} & x \neq 4 \\ 0 & x = 4 \end{cases}$$

Ans

L.H.L :—

$$\lim_{x \to 4^-} f(x)$$

$$\Rightarrow \lim_{x \to 4^-} \frac{|x-4|}{x-4}$$

$$\Rightarrow \lim_{x \to 4^-} \frac{x-4}{x-4}$$

Let,

$$x = 4-h$$

$$x \to 4, h \to 0$$

$$\lim_{h \to 0} \frac{|\cancel{4}-h-\cancel{4}|}{\cancel{4}-h-\cancel{4}}$$

$$\lim_{h \to 0} \frac{h}{-h} = -1$$

$$\lim_{h \to 0} -1 = -1$$

R.H.L :—

$$\lim_{x \to 4^+} f(x)$$

$$\Rightarrow \lim_{x \to 4^+} \frac{|x-4|}{x-4}$$

$$\Rightarrow \lim_{x \to 4^+} \frac{x-4}{x-4}$$

Let,

$$x = 4+h$$

$$x \to 4, h \to 0$$

$$\lim_{h \to 0} \frac{|\cancel{4}+h-\cancel{4}|}{\cancel{4}+h-\cancel{4}}$$

$$\lim_{h \to 0} \frac{h}{h} = 1$$

$$\lim_{h \to 0} 1 = 1$$

$\therefore$ L.H.L $\neq$ R.H.L.

Hence, $f(x)$ is not continuous.

$f(x)$ doesn't exists.

Definition of Differentiability :—

$f(x)$ is said to be differentiable at the point $x = a$ if the derivative $f'(a)$ exists at every point in its domain. It is given by

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

For a function to be differentiable at any point $x = a$ in its domain, it must be continuous at that particular point but vice-versa is not always true.

The derivatives of the basic trigonometric functions are;

1) $\frac{d}{dx}(\sin x) = \cos x$.

2) $\frac{d}{dx}(\cos x) = -\sin x$.

3) $\frac{d}{dx}(\tan x) = \sec^2 x$.

4) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

5) $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$.

6) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$.

Example :—

Consider the function $f(x) = (2x-3)^{\frac{1}{5}}$. Discuss its continuity and differentiability at $x = 3/2$.

Sol^n For checking the continuity, we need to check the left hand and right-hand limits and the value of the function at a point $x = a$.

L·H·L :— $\lim_{x \to a^-} f(x) = \lim_{x \to 3/2} (2x-3)^{\frac{1}{5}}$

$$= \left(2 \times \frac{3}{2} - 3\right)^{\frac{1}{5}}$$

$$= 0$$

R·H·L :— $\lim_{x \to a^+} f(x) = \lim_{x \to 3/2} (2x-3)^{\frac{1}{5}}$

$$= \left(2 \times \frac{3}{2} - 3\right)^{\frac{1}{5}}$$

$$= 0$$

$$L·H·L = R·H·L = f(a) = 0$$

Thus the function is continuous at about the point $x = 3/2$.

Now to chek differentiability at the given point, we know.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{f\left(\frac{3}{2}+h\right) - f\left(\frac{3}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\left[2\left(\frac{3}{2}\right)+h\right]-3\right)^{\frac{1}{5}} - \left(2\left(\frac{3}{2}\right)-3\right)^{\frac{1}{5}}}{h}$$

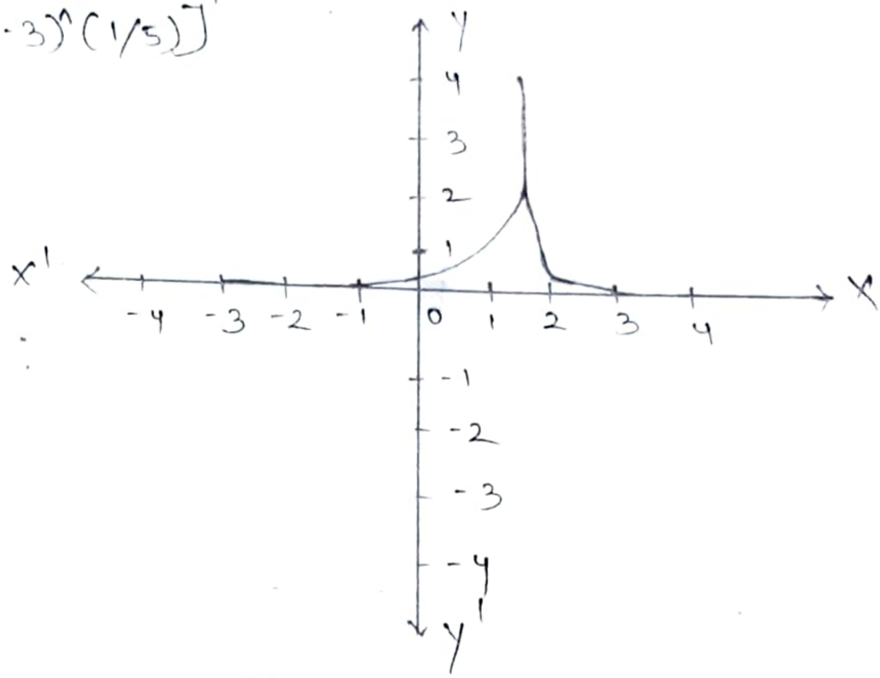
$$= \lim_{h \to 0} \frac{(\cancel{3}+2h-\cancel{3})^{\frac{1}{5}} - (\cancel{3}-\cancel{3})^{\frac{1}{5}}}{h}$$

$$= \lim_{h \to 0} \frac{(2h)^{\frac{1}{5}} - 0}{h}$$

$$= \lim_{h \to 0} \frac{2^{\frac{1}{5}}}{h^{\frac{4}{5}}} = \infty$$

Thus f is not differentiable at $x = 3/2$

$f'(x) = [(2x-3)^{\wedge}(1/5)]'$



We see that even though the function is continuous but it is not differentiable.

Rolle's Theorem :—

If a function defined on $[a,b]$ is

i) Continuous on $[a,b]$

ii) derivable on $]a,b[$ and

iii) $f(a) = f(b)$

There exist at least one point real number c between a and b such that $f'(c) = 0$.

Proof:— Let $[a,b]$ be a closed interval.

It is bounded.

Then m and M be infimum (g.l.b) and Supremum (l.u.b) of $f(x)$.

$\exists$ a point c and d on $[a,b]$.

$f(c) = m,\ f(d) = M$.

There are two possibility.

$m = M$ or $m \neq M$.

If $m = M$ over $[a,b]$

$f'(x) = 0\ \forall x \in [a,b]$

$m = M$ Then $f(a)$ and $f(b)$ are not equal.

$f(c) = m \neq f(a) \Rightarrow c \neq a$

$f(c) = m \neq f(b) \Rightarrow c \neq b$

This means c lies on $]a,b[$.

we shall show that

$f'(c) = 0$

$f'(c) < 0,\ f(x) < f(c) = m$

$f'(c) > 0,\ f(x) > f(c) = m$

which is Contradict.

Hence $f'(c) = 0$ (proved)

Example :-

No 1. Q. $f(x) = x^3 - 4x$ on $[-2, 2]$.

Soln Given,

$f(x) = x^3 - 4x$ on $[-2, 2]$

$f(-2) = (-2)^3 - 4(-2)$

$= -8 + 8 = 0$

$f(2) = 2^3 - 4 \cdot 2 = 8 - 8 = 0$

$f'(x) = 3x^2 - 4$.

$f'(c) = 3c^2 - 4$

$f'(c) = 0$

$\Rightarrow 3c^2 - 4 = 0$

$\Rightarrow 3c^2 = 4$

$\Rightarrow c^2 = 4/3$

$\Rightarrow c = \sqrt{4/3} = \pm \frac{2}{\sqrt{3}}$

Hence $c = \pm \frac{2}{\sqrt{3}}$ is a point.

lie on $[-2, 2]$

Hence the proved.

No 2. Q. $f(x) = (x-a)^m (x-b)^n$, where m and n are positive integers on $[a, b]$.

Soln Given, $f(x) = (x-a)^m (x-b)^n$

$f(a) = 0$

$f(b) = 0$

$f'(x) = (x-a)^m \cdot n(x-b)^{n-1} + (x-b)^n m(x-a)^{m-1}$

$= n(x-a)^m \cdot (x-b)^n \cdot (x-b)^{-1} + m(x-b)^n (x-a)^m \cdot (x-a)^{-1}$

$= (x-a)^m (x-b)^n [n(x-b)^{-1} + m(x-a)^{-1}]$.

$= (x-a)^m (x-b)^n \left(\frac{n}{x-b} + \frac{m}{x-a} \right)$

$f'(c) = (c-a)^m (c-b)^n \left[\frac{n}{c-b} + \frac{m}{c-a} \right]$

$f'(c) = 0$

$(c-a)^m (c-b)^n \left[\frac{n}{c-b} + \frac{m}{c-a} \right] = 0$

$(c-a)^m (c-b)^n = 0$

$(c-a)^m = 0$ $\quad$ $c-b = 0$

$c-a = 0$ $\quad$ $c = b$

$c = a$

No 3 Q. $f(x) = 1-(x-1)^{2/3}$ on $[0,2]$

Solⁿ. Given, $f(x) = 1-(x-1)^{2/3}$ on $[0,2]$

$f(0) = 1-(0-1)^{2/3}$

$= 1-(-1)^{2/3}$

$f(2) = 1-(2-1)^{2/3}$

$= 1-(1)^{2/3}$

$f'(x) = 0 - \frac{2}{3}(x-1)^{-1/3} \cdot 1$

$= -\frac{2}{3}(x-1)^{-1/3}$

$f'(c) = -\frac{2}{3}(c-1)^{-1/3}$

$f'(c) = 0$

$-\frac{2}{3}(c-1)^{-1/3} = 0$

$\Rightarrow (c-1)^{-1/3} = 0$

$\Rightarrow c-1 = 0$

$\Rightarrow c = 1$

CAUCHY MEAN VALUE THEOREM :—

If two function $f(x)$ and $g(x)$ defined on $[a,b]$ are,

i) Continuous on $[a,b]$.

ii) Derivable on $]a,b[$ and.

iii) $g'(x) \neq 0$, for any $x \in]a,b[$.

Then there exists at least one real number between a and b s.t.

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Proof

Consider the function.

$\phi(x) = f(x) + A g(x)$, $x \in [a,b]$.

A is a constant.

$\phi(a) = \phi(b)$

$$A = \frac{-f(x)}{g(x)} = \frac{-f(b)-f(a)}{g(b)-g(a)}$$

Now, function $\phi(x)$ is derivable over $f(x)$ and $g(x)$

i) Continuous on $[a,b]$.

ii) Derivable on $]a,b[$

iii) $\phi(a) = \phi(b)$

According to Rolle's Theorem.

$\exists$ a real number $c \in [a,b]$

s.t. $\phi'(c) = 0$

$\phi'(x) = f'(x) + A g'(x)$

$\phi'(c) = f'(c) + A g'(x)$

$$\frac{-f'(c)}{g'(c)} = A$$

$$\frac{-f'(c)}{g'(c)} = \frac{-f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)} \quad \text{(proved.)}$$

Example :—

No 1 Q. $f(x) = \sin x$.

$g(x) = \cos x$ on $[0, \pi]$.

Sol^n Given, $f(x) = \sin x$, $g(x) = \cos x$, on $[0, \pi]$

$f'(x) = \cos x$.

$g'(x) = -\sin x$.

$f(0) = \sin 0 = 0$

$f(\pi) = \sin \pi = 0$

$g(0) = \cos 0 = 1$

$g(\pi) = -1$

$f'(c) = \cos c$.

$g'(c) = -\sin c$.

$$\frac{\cos c}{\sin c} = \frac{0-0}{1+1}$$

$\Rightarrow -\cot c = 0$

$\Rightarrow c = \pi/2$

Hence $\pi/2 \in (0, \pi)$

No 2 Q. Examine cauchy mean value Theorem.

$f'(x) = 3x^2+1$

$g'(x) = 2x+1$

Soln

Given, $f'(x) = 3x^2+1$

$g'(x) = 2x+1$

$f'(1) = 3\cdot1^2+1 = 4$

$f'(2) = 3\cdot2^2+1 = 3\cdot4+1 = 12+1 = 13$

$g'(1) = 2\cdot1+1 = 3$

$g'(2) = 2\cdot2+1 = 4+1 = 5$

$f'(c) = 3c^2+1$

$g'(c) = 2c+1$

$$\frac{3c^2+1}{2c+1} = \frac{13-4}{5-3}$$

$$\Rightarrow \frac{3c^2+1}{2c+1} = \frac{9}{2}$$

$\Rightarrow 6c^2+2 = 18c+9$

$\Rightarrow 6c^2-18c+2-9=0$

$\Rightarrow 6c^2-18c-7=0$

$$\Rightarrow \frac{18\pm\sqrt{(18)^2-4\cdot6\cdot(-7)}}{2\cdot6} = 0$$

$$\Rightarrow \frac{18\pm\sqrt{324+168}}{12} = 0$$

$\Rightarrow 18\pm\sqrt{492} = 0.$

LAGRANGE MEAN VALUE THEOREM :—

If a function $f(x)$ is defined on $[a,b]$ is

i) Continuous on $[a,b]$ and.

ii) Derivable on $]a,b[$.

Then ∃ at least a real number c between a and b

s.t.

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

Proof Let us consider a function.

$$\phi(x) = f(x) + A(x), \ x \in [a,b]$$

where A is a constant.

$$\phi(a) = \phi(b)$$

$$\therefore A = -\frac{f(b)-f(a)}{b-a}.$$

Now,

$\phi(x)$ be too continuous and derivable

i) Continuous on $[a,b]$.

ii) derivable on $]a,b[$.

iii) $\phi(a) = \phi(b)$

According to Rolle's Theᵐ.

∃ a real number $c \in [a,b]$. s.t.

$$\phi'(c) = 0$$

$$\phi'(x) = f'(x) + A.$$

$$\Rightarrow A = -f'(x)$$

$$\Rightarrow A = -f'(c)$$

$$\Rightarrow -f'(c) = \frac{-f(b)-f(a)}{b-a}.$$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a} \quad \text{(proved)}$$

Example.

No 1 Q. $f(x) = 2x^2 - 7x + 10$ on $[2, 5]$.

Soln $f'(x) = 4x - 7$.

$f'(c) = 4c - 7$.

$f(5) = 2\cdot5^2 - 7\cdot5 + 10$

$= 2\cdot25 - 35 + 10$

$= 50 - 35 + 10$

$= 60 - 35 = 25$.

$f(2) = 2\cdot2^2 - 7\cdot2 + 10$

$= 8 - 14 + 10$

$= 18 - 14$

$= 4$

$\frac{25-4}{5-2} = 4c - 7$.

$\Rightarrow \frac{21}{3} = 4c - 7$

$\Rightarrow 7 = 4c - 7$.

$\Rightarrow 7 + 7 = 4c$

$\Rightarrow 14 = 4c$

$\Rightarrow c = 14/4 = 7/2$

Hence, $7/2 \in [2, 5]$.

No 2 Q. Examine Langrange mean value Thm

$f(x) = x(x-1)(x-2)$ on $(0, 1/2)$

Soln Given,

$f(x) = x(x-1)(x-2)$ on $[0, 1/2]$.

$f(x) = x^3 - 3x^2 + 2x$.

$f'(c) = \frac{f(b) - f(a)}{b-a}$.

$f'(x) = 3x^2 - 6x + 2$.

$f'(c) = 3c^2 - 6c + 2$

$f(0) = 0$

$f(1/2) = (1/2)^3 - 3\cdot(1/2)^2 + 2\times 1/2$

$= 1/8 - 3\cdot 1/4 + 1$

$= 1/8 + 1 - 3/4$

$= 9/8 - 3/4$

$= \frac{9-6}{8} = 3/8$

$$3c^2 - 6c + 2 = \frac{3/8 - 0}{1/2 - 0}$$

$$\Rightarrow 3c^2 - 6c + 2 = \frac{3/8}{1/2}$$

$\Rightarrow 3c^2 - 6c + 2 = 3/4$

$\Rightarrow 3c^2 - 6c + 2 - 3/4 = 0$

$\Rightarrow 3c^2 - 6c + \frac{8-3}{4} = 0$

$\Rightarrow 3c^2 - 6c + 5/4 = 0$

$\Rightarrow 12c^2 - 24c + 5 = 0$

$$c = \frac{24 \pm \sqrt{(-24)^2 - 4\cdot 12\cdot 5}}{2\cdot 12}$$

$$= \frac{24 \pm \sqrt{576 - 240}}{24}$$

$$= \frac{24 \pm \sqrt{336}}{24}$$

$$= \frac{24 \pm 4\sqrt{21}}{24}$$

$$= \frac{4(6 + \sqrt{21})}{\cancel{24}} = \frac{6 \pm \sqrt{21}}{6}$$

$\alpha = \frac{6 \pm \sqrt{21}}{6}$, $\beta = \frac{6 - \sqrt{21}}{6}$

So $\frac{6 - \sqrt{21}}{6} \in [0, 1/2]$

Maclaurin Series :—

Putting $a=0$ in 2nd form of Taylor's theorem we get :

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots\cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0)$$

$$+ \frac{x^n (1-\theta)^{n-p}}{p(n-1)!} f^n(\theta x)$$

called maclaurin's theorem.

Example :—

No 1 Q. $\sin x$.

Soln Given, $f(x) = \sin x$. $\Rightarrow f(0) = 0$

$f'(x) = \cos x$. $\Rightarrow f'(0) = 1$

$f''(x) = -\sin x$ $\Rightarrow f''(0) = 0$

$f'''(x) = -\cos x$ $\Rightarrow f'''(0) = -1$

$$f(x) = f(0) + x \cdot \frac{f'(0)}{1!} + x^2 \cdot \frac{f''(0)}{2!} + \cdots\cdots$$

$$= 0 + x \cdot \frac{1}{1!} + x^2 \cdot \frac{0}{2!} + x^3 \cdot \frac{-1}{3!} \cdots\cdots$$

$$= 0 + x \cdot \frac{1}{1} + x^2 \cdot \frac{0}{2} + x^3 \frac{1}{6} \cdots\cdots$$

$$= x + \frac{x^3}{6} \cdots\cdots$$

No 2 Q. e^x

Soln Given, $f(x) = e^x$, $\Rightarrow f(0) = 1$

$f'(x) = e^x$ $\Rightarrow f'(0) = 1$

$f''(x) = e^x$ $\Rightarrow f''(0) = 1$

$f'''(x) = e^x$ $\Rightarrow f'''(0) = 1$

$$f(x) = 1 + x \cdot \frac{1}{1!} + x^2 \cdot \frac{1}{2!} + x^3 \cdot \frac{1}{3!} + \cdots$$

$$= 1 + x \cdot \frac{1}{1} + x^2 \cdot 1/2 + x^3 \cdot 1/6 + \cdots$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

No 3 Q. $\log(1+x)$

Soln $f(x) = \log(1+x)$ $\quad f(0) = 1$

$f'(x) = \frac{1}{1+x}$ $\quad f(0) = 1$

$f''(x) = \frac{1}{(1+x)^2}$ $\quad f''(0) = 1$

$f^n(x) = \frac{1}{(1+x)^n}$ $\quad f^n(0) = 1$

Now from maclaurin theorem.

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^n(0)$$

$$\log(1+x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} f^n(0) + \cdots$$

Taylor's Theorem :—

From of remainder after nth term :—

The term $R_n = \frac{R^n (1-p)^{n-p}}{p(n-1)!} f^n(a + \theta h)$

is called Taylor's remainder after nth term.

The teorem with this remainder is called Taylor's theorem with schlomilch and Roche form of remainder.

ii) For $p = 1$

$R_n = \frac{h^n (1-b)^{n-1}}{(n-1)!} f^n(a+\theta h)$ is called cauchy's form of remainder.

2nd form of Taylor theorem :—

If f satisfy condition of Taylor's theorem in $(a, a+h)$ and $x \in (a, a+h)$

Then,

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a)$$

$$+ \frac{(x-a)^n}{p(n-1)!} (1-a)^{n-p} \, f^n(a+\theta h) \, \theta \, (x-a)$$

is called 2nd form of Taylor's theorem.

Example :—

No 1. Q. $f(x) = \sin x$ at $x = \pi/2$

Soln Here,

Given,

$f(x) = \sin x, \quad f(\pi/2) = \sin \pi/2 = 1$

$f'(x) = \cos x, \quad f'(\pi/2) = \cos \pi/2 = 0$

$f''(x) = -\sin x, \quad f''(\pi/2) = -\sin \pi/2 = -1$

$f'''(x) = -\cos x, \quad f'''(\pi/2) = -\cos \pi/2 = 0$

$$f(x) = f(a) + f'(a) \frac{(x-a)^1}{1!} + f''(a) \frac{(x-a)^2}{2!} + \cdots\cdots + f^n(a) \frac{(x-a)^n}{n!}$$

$$= 1 + 0 \cdot \frac{x - \pi/2}{1!} + (-1) \frac{(x - \pi/2)^2}{2!} + 0 \frac{(x - \pi/2)^3}{3!} + \cdots$$

$$= 1 + 0 - \frac{(x - \pi/2)^2}{2} + \frac{(x - \pi/2)^4}{24}$$

$\frac{No}{2}$ Q. $f(x) = \tan x$ at $x = 0$

Solⁿ Here, Given,

$f(x) = \tan x, \quad f(0) = \tan 0 = 0$

$f'(x) = \sec^2 x, \quad f(0) = \sec^2 0 = 1$

$f''(x) = 2\sec x \cdot \sec x \cdot \tan x. \quad f''(0) = 0$

$= 2\sec^2 x \tan x$

$f'''(x) = 2\sec 2x \cdot \sec^2 x + 2\tan x \cdot 2\sec x \cdot \sec x \cdot \tan x.$

$= 2\sec^4 x + 4\sec^2 x \tan^2 x.$

$f'''(0) = 2$

$f(x) = 0 + 1 \cdot \frac{(x-0)^1}{1!} + 0 \frac{(x-a)^2}{2!} + 2 \cdot \frac{(x-0)^3}{3!} + \cdots$

$= x + \frac{x^3}{3} + \cdots$

L-Hospital's Rule :—

Sometimes we unable to evaluate limit in such exceptional cases is know as L-Hospital's Rule.

L-Hospital's Rule allow the evaluation of indeterminate form.

Like $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \times \infty, 1^\infty, \infty^0, 0^0$.

i) $\frac{0}{0}$ form.

$$\frac{\lim_{x \to 0} f(x)}{\lim_{x \to 0} g(x)} = \frac{0}{0} \text{ form.}$$

If $\lim_{x \to a} f(x) = 0, \lim_{x \to a} g(x) = 0$

Ex $\lim_{x \to 0} \frac{e^x - \cos x}{x}$

$\frac{0}{0}$ form.

(ii) $\frac{\infty}{\infty}$ form.

$\lim_{x \to \infty} \frac{e^x}{\ln x} \quad \frac{\infty}{\infty}$ form.

L-Hospital's Rule for $\frac{0}{0}$ form.

Let $f(x)$ and $g(x)$ be two differentiable functions $g'(x) \neq 0$.

$\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$

Then,

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = l \text{ is finite:}$$

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = l.$

(1) $\frac{0}{0}$ form :—

Example-1

$$\lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x^2 \sin x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x^3} \cdot \frac{x}{\sin x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{\frac{d}{dx}(e^x - e^{-x} - 2x)}{\frac{d}{dx} x^3} \times \lim_{x \to 0} \frac{\frac{d}{dx} x}{\frac{d}{dx} \sin x}.$$

(L-Hospital rule)

$$= \lim_{x \to 0} \frac{e^x + e^{-x} - 2}{3x^2} \times \lim_{x \to 0} \frac{1}{\cos x}.$$

$$= \lim_{x \to 0} \frac{e^x + e^{-x} - 2}{3x^2} \times \frac{1}{1}$$

$$= \lim_{x \to 0} \frac{e^x + e^{-x} - 2}{3x^2} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{\frac{d}{dx}(e^x + e^{-x} - 2)}{\frac{d}{dx} 3x^2} \quad (\text{L'Hospital rule})$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x}}{6x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x}}{6}$$

$$= \frac{2}{6} = \frac{1}{3}$$

(2) $\frac{\infty}{\infty}$ form :—

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$ and if $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists then $\lim_{x \to a} \frac{f(x)}{g(x)}$ also exists and $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Example-1

$$\lim_{x \to \infty} \frac{x^4}{e^x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to \infty} \frac{\frac{d}{dx} x^4}{\frac{d}{dx} e^x} \quad (\text{L'Hospital Rule})$$

$$= \lim_{x \to \infty} \frac{4x^3}{e^x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to \infty} \frac{\frac{d}{dx} 4x^3}{\frac{d}{dx} e^x} \quad \text{(L' Hospital Rule)}$$

$$= \lim_{x \to \infty} \frac{12x^3}{e^x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to \infty} \frac{\frac{d}{dx} 12x^2}{\frac{d}{dx} e^x} \quad \text{(L' Hospital Rule)}$$

$$= \lim_{x \to \infty} \frac{24x}{e^x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to \infty} \frac{24}{e^x} = 0$$

③ $0 \times \infty$ form :—

Consider $\lim_{x \to 0} \{f(x) \cdot g(x)\}$

Let $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$.

Then $\lim_{x \to a} \{f(x) - g(x)\} = \lim_{x \to a} \frac{f(x)}{\left\{\frac{1}{g(x)}\right\}}$ or

$$\lim_{x \to a} \frac{g(x)}{\left\{\frac{1}{f(x)}\right\}}$$

Thus $\lim_{x \to 0} \{f(x) \cdot g(x)\}$ is reduced to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Hence L'Hospital's Rule is applied to find the limit.

Example :—

$$\lim_{x\to 0} x^2 \log x^2 \quad (0\times\infty \text{ form})$$

$$= \lim_{x\to 0} \frac{\log x^2}{\left(\frac{1}{x^2}\right)} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\frac{1}{x^2}\cdot 2x}{-\frac{2}{x^3}}$$

$$= \lim_{x\to 0} (-x^2) = 0 .$$

(4) $\infty-\infty$ form :—

Let $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to 0} g(x) = \infty$.

Then $\lim_{x\to a} \{f(x) - g(x)\}$

$$\lim_{x\to a} \left\{ \frac{1}{\frac{1}{f(x)}} - \frac{1}{\frac{1}{g(x)}} \right\} = \lim_{x\to a} \frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x)}\cdot\frac{1}{g(x)}}$$

$\left(\frac{0}{0} \text{ form}\right)$

This can be evaluated by using L'Hospital's Rule.

Example :—

$$\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right) \quad (\infty-\infty \text{ form})$$

$$= \lim_{x\to 0} \frac{\sin^2 x - x^2}{x^2 \sin^2 x} .$$

$$= \lim_{x\to 0} \left(\frac{\sin^2 x - x^2}{x^4} \cdot \frac{x^2}{\sin^2 x}\right).$$

$$= \lim_{x \to 0} \frac{\sin^2 x - x^2}{x^4} \cdot \lim_{x \to 0} \left(\frac{x}{\sin x}\right)^2$$

$$= \lim_{x \to 0} \frac{\sin^2 x - x^2}{x^4} \left(\frac{0}{0} \text{ form}\right) \left[\because \lim_{x \to 0} \left(\frac{x}{\sin x}\right)^2 = 1\right]$$

$$= \lim_{x \to 0} \frac{2 \sin x \cdot \cos x - 2x}{4x^3}$$

$$= \lim_{x \to 0} \frac{\sin 2x - 2x}{4x^3} \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{2 \cos 2x - 2}{12 x^2} \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{-4 \sin 2x}{24x} \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{-8 \cos 2x}{24} = \frac{-8}{24} = -1/3$$

(5) $0^0, 1^\infty, \infty^0$ form :—

Consider the form 0^0

Let $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$

Then $\lim_{x \to a} [f(x)]^{g(x)}$ takes the form 0^0.

Let $k = \lim_{x \to a} [f(x)]^{g(x)}$

$$\Rightarrow \log k = \log \left[\lim_{x \to a} [f(x)]^{g(x)}\right]$$

$$= \lim_{x \to a} \log [f(x)]^{g(x)}$$

$$= \lim_{x \to a} g(x) \log f(x) \quad (0 \times \infty \text{ form})$$

This can be reduced to $\frac{0}{0}$ form or $\frac{\infty}{\infty}$ form and its limit is evaluated by L'Hospital Rule.

Suppose limit $= l$.

$\therefore \log k = l \Rightarrow k = e^{l}$

Similarly, limit of other two forms can be evaluated.

Example :—

$$\lim_{x \to 1} x^{\frac{1}{1-x}}$$

Let $k = \lim_{x \to 1} x^{\frac{1}{1-x}}$

$$\Rightarrow \log k = \lim_{x \to 1} \frac{1}{1-x} \log x .$$

$$\Rightarrow \lim_{x \to 1} \frac{\log x}{1-x} \left(\frac{0}{0} \text{ form}\right)$$

$$\Rightarrow \lim_{x \to 1} \left(\frac{\frac{1}{x}}{-1}\right) = -1$$

$$\Rightarrow k = e^{-1}$$